E and j vary discontinuously, but it might be possible to use numerical solutions as a basis for an integral method, as difficult boundary conditions have been handled in other problems.

References

[1] Dahlberg, E., "On the one-dimensional flow of a conducting gas in crossed fields," Quart. Appl. Math. **XIX,** 177–193 (1961).

[2] Kerrebrock, J. L., "Electrode boundary layers in direct current plasma accelerators," J. Aerospace Sci. **28,** 631–643 (1961).

[3] Hale, F. J., "Insulator boundary layers in magnetohydrodynamic channels," Mass. Inst. Tech., Ph.D. Thesis, Dept. Aeronaut. and Astronaut. (1962).

[4] Moffatt, W. C., "Boundary layer effects in magnetohydrodynamic flows," Magnetogasdynamic Lab., Dept. Mech. Eng., Mass. Inst. Tech. Rept. 61-4 (1961).

[5] Resler, E. L., Jr. and Sears, W. R., "The prospects for magneto-aero-dynamics," J. Aeronaut. Sci. **25,** 235–248 (1958).

[6] Culick, F. E. C., "Magnetogasdynamic channel flow and convective heat transfer," Calif. Inst. Tech., Daniel and Florence Guggenheim Jet Propulsion Center, TN 6 (1962).

Procedure for the Determination of Impact Probabilities

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The various steps required to determine the distribution of hits around a two-dimensional or three-dimensional target caused by random normal errors in the fire-control parameters are discussed. A method is presented to express the distribution as a function of miss distance rather than of the three space variables.

Nomenclature

a,b,c	= distribution parameters
d_{ij}	= elements of D
D	= n by m matrix of partial derivatives $\partial f_i/\partial s_j$
D'	= transposed D matrix
f_i	= functions of fire-control parameters
I	= integral
m	= number of fire-control parameters
n	= number of dimensions, 2 or 3
p	= probability density function
P	= a transformation matrix that yields principal axes
P'	= transposed P matrix
r	= miss distance
s_i	= fire-control parameters
T	= $\mu_2{}^2/\mu_4$
v_{ij}	= elements of V_s
V_s	= m by m variance-covariance matrix of fire-control parameters
V_x	= n by n variance-covariance matrix of impact coordinates
$V_x{}'$	= variance-covariance matrix along principal axes
x_i	= ith impact coordinate
δs_j	= error in jth fire-control parameter
δx_i	= ith component of miss distance
$\delta x_i{}' \equiv \delta_i$	= miss distances along principal axis
$\langle \Delta X \rangle$	= n by 1 matrix containing δx_i
$\langle \Delta S \rangle$	= m by 1 matrix containing δs_j
$\mu_k(r)$	= kth moment of r about origin
$\sigma_{s_j}{}^2$	= variance of jth fire-control parameter
$\sigma_{x_i'}{}^2 \equiv \sigma_i{}^2$	= the eigenvalues of V_x = the diagonal elements of $V_x{}'$ = the variances along principal axes

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Introduction

THE impact coordinates of a missile as a function of fire-control parameters may be predicted by equations of the following form:

$$x_i = f_i(s_1, s_2, \ldots, s_m) \qquad 1 \leq i \leq n \tag{1}$$

where s_j represents any number of parameters, such as the initial coordinates, velocities, time, etc. Depending on the dimensionality of the target, n may be two or three.

A variation in the impact coordinates δx_i caused by small variations in the parameters δs_j may be expressed by the following equation:

$$\langle \Delta X \rangle = D \langle \Delta S \rangle \tag{2}$$

where D is an n by m matrix, the elements of which are defined by

$$d_{ij} = \partial f_i / \partial s_j \tag{3}$$

Random variations in the parameters are usually assumed to be normally and independently distributed with mean 0 and standard deviation σ_{s_j}. Associated with these random variations is a "variance-covariance" matrix[1] v_s which is of dimensions m by m and has elements defined by

$$\begin{aligned} v_{ij} &= 0 \qquad & i &\neq j \\ v_{ij} &= \sigma_{s_j}{}^2 \qquad & i &= j \end{aligned} \tag{4}$$

The variance-covariance matrix for the impact coordinates is then found by a simple matrix product

$$V_x = D V_s D' \tag{5}$$

where V_x is a symmetrical n by n matrix.

V_x is the matrix of second moments of the probability density function in the reference frame of the impact coordinates, and in general, off-diagonal terms will appear. It is well known in both statistics and mechanics, where such matrices appear, that there exists a reference frame where the off-diagonal terms will not appear. The transformation equation takes the same form as Eq. 5.[2]

$$V_x = P V_{x'} P' \tag{6}$$

where $V_{x'}$ is a diagonal matrix and P rotates the original axes into principal axes.

The diagonal elements of $V_{x'}$ are the eigenvalues of V_x and may be found by various techniques such as are described in Todd.[3] It is not necessary to find the P matrix.

The elements $\sigma_{x_1'}{}^2$, $\sigma_{x_2'}{}^2$, and $\sigma_{x_3'}{}^2$ (from $V_{x'}$) are the variances along the principal axes and completely describe the probability distribution of hits around the target under the forementioned assumptions.

A simplification in notation is now in order. Let $\sigma_{x_i'}{}^2 \equiv \sigma_i{}^2$ and let the ith component of miss distance along the principal axes be denoted δ_i instead of $\delta x_i{}'$.

The distribution of hits around the target may now be written for the three-dimensional situation:

$$p(\delta_1,\delta_2,\delta_3) = a \exp\{-\tfrac{1}{2}[(\delta_1/\sigma_1)^2 + (\delta_2/\sigma_2)^2 + (\delta_3/\sigma_3)^2]\} \tag{7}$$

where a is the normalizing factor that gives a unit volume under the curve.

Analytically, the problem is completely solved but not in a practical form on which to base human judgment, for in practice one is interested only in how far the target is missed and will have no information as to the direction of the miss. So this information must be deleted from the probability density function above.

The direct approach is to convert to polar or spherical coordinates and integrate out the angular dependence. The integration is possible analytically only in the special situations where all the variances are equal. The distributions obtained in these situations are

$$p(r) = ar \exp(-r^2/2\sigma^2) \quad \text{(two-dimensional situation)} \quad (8)$$

$$p(r) = ar^2 \exp(-r^2/2\sigma^2) \quad \text{(three-dimensional situation)} \quad (9)$$

where $\sigma = \sigma_1 = \sigma_2 =$ (in the three-dimensional situation) σ_3. When the variances are not equal the process must be done numerically. This has been done for a wide range of points in the two-dimensional situation by Baur.[4]

Another approach is presented here. It was developed to fulfill the following objectives: 1) the method should have a relative error less than 1%; 2) it should work for both the two-dimensional and three-dimensional situations; 3) it should form the basis of a fast, efficient subroutine for digital computers; 4) it should provide the r probability distribution rather than just the points thereof.

Method

The method is based on the fact that certain moments of the distribution [Eq. (7)] may be obtained analytically. The r moments may be defined as follows:

$$\mu_k(r) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r^k f(\delta_1,\delta_2,\delta_3)d(\delta_1)d(\delta_2)d(\delta_3) \quad (10)$$

where $r^2 = (\delta_1)^2 + (\delta_2)^2 + (\delta_3)^2$

Integration by parts yields the even moments of the distribution, and, in particular, the following:

$$\mu_2 = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)\mu_0 \quad (11)$$

$$\mu_4 = [3(\sigma_1^4 + \sigma_2^4 + \sigma_3^4) + 2(\sigma_1^2\sigma_2^2 + \sigma_1^2\sigma_3^2 + \sigma_2^2\sigma_3^2)]\mu_0 \quad (12)$$

where μ_0 is arbitrary and has already been taken to be unity.

Let us assume that the distribution as a function of miss distance (r) always takes the form of Eqs. (8) and (9), namely

$$p(r) = ar^b \exp(-cr^2) \quad (13)$$

Then one may determine the parameters a, b, and c so that the 0th second and fourth moments of this distribution are the same as those of the distribution of Eq. (7). Integration by parts will also give the even moments of this distribution [Eq. (13)].

$$\mu_2 = [(b + 1)/2c]\mu_0 \quad (14)$$

$$\mu_4 = [(b + 3)/2c]\mu_2 \quad (15)$$

where again μ_0 is taken as unity.

Then b and c are found as follows:

$$b = (3T - 1)/(1 - T) \quad (16)$$

$$c = (b + 1)/2\mu_2 \quad (17)$$

where $T = \mu_2^2/\mu_4$ and the numerical values of μ_2 and μ_4 are taken from Eqs. (11) and (12). The normalizing parameter a is found by evaluating the integral

$$I = \int_0^\infty r^b \exp(-cr^2)dr$$

which may be expanded as follows:

$$I = \frac{r^{b+1}}{b+1} - \frac{cr^{b+3}}{b+3} + \frac{1}{2}\left(\frac{c^2r^{b+5}}{b+5}\right) - \frac{1}{3!}\left(\frac{c^3r^{b+7}}{b+7}\right) + \ldots \quad (18)$$

A value of $r = 5/(2c)^{1/2}$ was found to be sufficiently close to infinity to evaluate the total integral.

Finally,

$$a = 1/I \quad (19)$$

Once the parameters a, b, and c are found, the expansion [Eq. (18)] may be used to find the "cumulative" probability.

Results

The distributions obtained by this method were found to have errors of less than 0.5%, based on the error in μ_i that was computed numerically for a number of test cases. In the two-dimensional case the agreement with Baur is within 1%. Four examples of the distributions obtained by this method are given for a two-dimensional target (Fig. 1). These are obtained by setting one of the variances (σ_3^2) to zero in the foregoing equation. No figures appear for the three-dimensional case because covering an interesting range of values for the three variances requires too many figures for an article of this size.

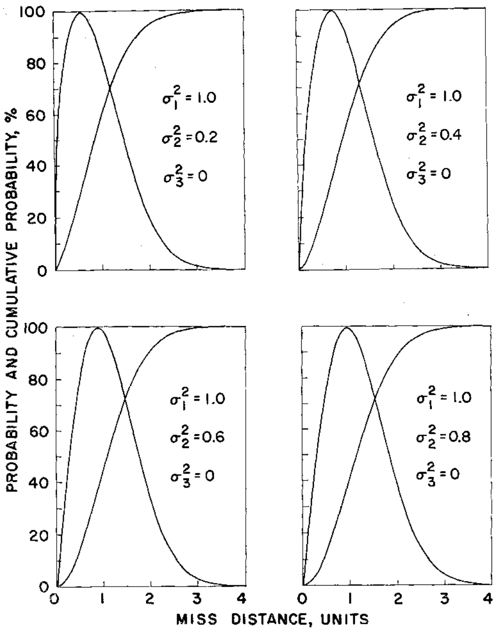

Fig. 1 Distributions obtained for a two-dimensional target.

All calculations described above have been coded in the FORTRAN language and are available from the author.

References

[1] Kempthorne, O., *The Design and Analysis of Experiments* (John Wiley and Sons, Inc., New York, 1952), p. 55.

[2] Ref. 1, pp. 65ff.

[3] Todd, J., *A Survey of Numerical Analysis* (McGraw-Hill Book Co., New York, 1962), p. 243ff.

[4] Baur, E. H., "On the transformation of probability ellipses into circles for impact problems," Land-Air, Inc., Point Mugu, Calif., Rept. 34 (1962).

Low Pressure Rocket Extinction

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Introduction

THERE has been considerable interest recently in designing solid rockets for low pressure operation because of the shell weight economy this would allow. Two primary requirements for this type of design are that there be no loss of combustion efficiency and that the rocket ignite

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